



# The complexity of equilibria: Hardness results for economies via a correspondence with games<sup>☆</sup>

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## ABSTRACT

We give a reduction from any two-player game to a special case of the Leontief exchange economy, with the property that the Nash equilibria of the game and the equilibria of the market are in one-to-one correspondence.

Our reduction exposes a computational hurdle inherent in solving certain families of market equilibrium problems: finding an equilibrium for Leontief economies is at least as hard as finding a Nash equilibrium for two-player nonzero sum games, a problem recently proven to be PPAD-complete.

As a corollary of the one-to-one correspondence, we obtain a number of hardness results for questions related to the computation of market equilibria, using results already established for games [I. Gilboa, E. Zemel, Nash and correlated equilibria: Some complexity considerations, *Games and Economic Behavior* 1 (1989) 80–93]. In particular, among other results, we show that it is NP-hard to say whether a particular family of Leontief exchange economies, that is guaranteed to have at least one equilibrium, has more than one equilibrium.

Perhaps more importantly, we also prove that it is NP-hard to decide whether a Leontief exchange economy has an equilibrium. This fact should be contrasted against the known PPAD-completeness result of [C.H. Papadimitriou, On the complexity of the parity argument and other inefficient proofs of existence, *Journal of Computer and System Sciences* 48 (1994) 498–532], which holds when the problem satisfies some standard sufficient conditions that make it equivalent to the computational version of Brouwer's Fixed Point Theorem.

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## 1. Introduction

In a *strategic game*, each player takes decisions which depend on the strategies available to the other players. The *exchange economy* setting differs from the scenario of games, because the equilibrium prices have the “decentralizing” effect of making the strategic decisions of the economic agents independent. However there is a natural interplay between *Walrasian equilibria* for exchange economies and *Nash equilibria* for games: one of the very first proofs of existence of an economic equilibrium is built upon the existence of a Nash equilibrium in an associated abstract game. The actors in this game are the economic agents and an extra player, the market, whose strategy set coincides with the prices.

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In this paper we first establish a one-to-one correspondence between the Nash equilibria in any two-player nonzero sum game and the Walrasian equilibria in an associated exchange economy. We then use this correspondence to state a number of hardness results for economies.

### Bimatrix games

We consider two-player games in *normal form*, also known as *bimatrix games*. These games can in fact be described by a pair  $(A, B)$  of matrices, whose entries represent the *payoffs* of the two players, called row and column player.  $A = (a_{ij})$  (resp.  $B = (b_{ij})$ ) is the payoff matrix of the row (resp. column) player.

The rows (resp. columns) of  $A$  and  $B$  are indexed by the row (resp. column) player's *pure strategies*.

The entry  $a_{ij}$  is the payoff to the row player, when she plays her  $i$ -th pure strategy and the opponent plays his  $j$ -th pure strategy. Similarly,  $b_{ij}$  is the payoff to the column player, when he plays his  $j$ -th pure strategy and the opponent plays her  $i$ -th pure strategy.

A *mixed strategy* is a probability distribution over the set of pure strategies. In a mixed strategy a player associates to her  $i$ -th pure strategy a quantity  $p_i$  between 0 and 1, such that  $\sum_i p_i = 1$ , where the sum ranges over all the pure strategies.

Let us consider the game  $(A, B)$ , where  $A$  and  $B$  are  $m \times n$  matrices. In such a game the row player has  $m$  pure strategies, while the column player has  $n$  pure strategies. Let  $x$  (resp.  $y$ ) be a mixed strategy of the row (resp. column) player. Strategy  $x$  is the  $m$ -tuple  $x = (x_1, x_2, \dots, x_m)$ , where  $x_i \geq 0$ , and  $\sum_{i=1}^m x_i = 1$ . Similarly,  $y = (y_1, y_2, \dots, y_n)$ , where  $y_i \geq 0$ , and  $\sum_{i=1}^n y_i = 1$ .

When the pair of mixed strategies  $x$  and  $y$  is played, the entry  $a_{ij}$  contributes to the *expected payoff* of the row player with weight  $x_i y_j$ . The expected payoff of the row player can be evaluated by adding up all the entries of  $A$  weighted by the corresponding entries of  $x$  and  $y$ , i.e.  $\sum_{ij} x_i y_j a_{ij}$ . This can be rewritten as  $\sum_i x_i \sum_j a_{ij} y_j$ , which can be expressed in matrix terms as  $x^T A y$ . Similarly, the expected payoff of the column player is  $x^T B y$ .

A pair of mixed strategies  $(x, y)$  is in Nash equilibrium if  $x^T A y \geq x'^T A y$ , for all stochastic  $m$ -vectors  $x'$ , and  $x^T B y \geq x^T B y'$ , for all stochastic  $n$ -vectors  $y'$ .

We say that  $x$  (resp.  $y$ ) is a *Nash equilibrium strategy* for the row (resp. column) player.

The set of indices such that  $x_i > 0$  (resp.  $y_i > 0$ ) is called the *support* of the Nash equilibrium strategy  $x$  (resp.  $y$ ).

### Exchange economies

We now describe the model of an *exchange economy*, and define the appropriate notion of equilibrium.

Let us consider  $m$  economic agents interested in trading  $n$  goods. Let  $\mathbf{R}_+^n$  denote the subset of  $\mathbf{R}^n$  with all nonnegative coordinates. The  $j$ -th coordinate in  $\mathbf{R}^n$  will stand for good  $j$ . Each trader  $i$  has a concave utility function  $u_i : \mathbf{R}_+^n \rightarrow \mathbf{R}_+$ , which represents her preferences for the different bundles of goods, and an initial endowment of goods  $w_i = (w_{i1}, \dots, w_{in}) \in \mathbf{R}_+^n$ . At given prices  $\pi \in \mathbf{R}_+^n$ , trader  $i$  will sell her endowment, and get the bundle of goods  $x_i = (x_{i1}, \dots, x_{in}) \in \mathbf{R}_+^n$  which maximizes  $u_i(x)$  subject to the budget constraint<sup>2</sup>  $\pi \cdot x \leq \pi \cdot w_i$ .

An equilibrium for an exchange economy is a vector of prices  $\pi = (\pi_1, \dots, \pi_n) \in \mathbf{R}_+^n$  at which, for each trader  $i$ , there is a bundle  $\bar{x}_i = (\bar{x}_{i1}, \dots, \bar{x}_{in}) \in \mathbf{R}_+^n$  of goods such that the following two conditions hold:

1. For each trader  $i$ , the vector  $\bar{x}_i$  maximizes  $u_i(x)$  subject to the constraints  $\pi \cdot x \leq \pi \cdot w_i$  and  $x \in \mathbf{R}_+^n$ .
2. For each good  $j$ ,  $\sum_i \bar{x}_{ij} \leq \sum_i w_{ij}$ .

For any price vector  $\pi$ , the vector  $x_i(\pi)$  that maximizes  $u_i(x)$  subject to the constraints  $\pi \cdot x \leq \pi \cdot w_i$  and  $x \in \mathbf{R}_+^n$  is called the *demand* of trader  $i$  at prices  $\pi$ .

Then  $X^k(\pi) = \sum_i x_{ik}(\pi)$  denotes the *market demand* of good  $k$  at prices  $\pi$ , and  $Z^k(\pi) = X^k(\pi) - \sum_i w_{ik}$  the *market excess demand* of good  $k$  at prices  $\pi$ . The vectors  $X(\pi) = (X^1(\pi), \dots, X^n(\pi))$  and  $Z(\pi) = (Z^1(\pi), \dots, Z^n(\pi))$  are called *market demand* (or aggregate demand) and *market excess demand*, respectively.

We say that the market satisfies *Walras' Law* if for any price  $\pi$ , we have  $\pi \cdot Z(\pi) = 0$ .

When the market is described in terms of the aggregate excess demand function, the equilibrium conditions are satisfied by a vector of prices  $\pi = (\pi_1, \dots, \pi_n) \in \mathbf{R}_+^n$  such that  $Z(\pi)$  is well-defined and  $Z^j(\pi) \leq 0$ , for each  $j$ .

The requirement that  $Z(\pi)$  is well defined means that, for each trader  $i$ , there is a vector  $\bar{x}_i \in \mathbf{R}_+^n$  such that  $\pi \cdot \bar{x}_i \leq \pi \cdot w_i$  and  $u_i(x) \leq u_i(\bar{x}_i)$  for every  $x \in \mathbf{R}_+^n$  such that  $\pi \cdot x \leq \pi \cdot w_i$ . In other words, each trader has an optimal bundle among her feasible bundles.

Sometimes, one assumes that the traders maximize some special utility functions. Typical examples are linear, Cobb–Douglas, Leontief, and CES functions.

A linear utility function has the form  $u_i(x) = \sum_j a_{ij} x_{ij}$ . The Cobb–Douglas function has the form  $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$ , where  $\sum_j a_{ij} = 1$ . The Leontief function has the form  $u_i(x) = \min_j a_{ij} x_{ij}$ . A CES (constant elasticity of substitution) utility function

<sup>1</sup> We use the notation  $x^T$  to denote the transpose of vector  $x$ .

<sup>2</sup> Given two vectors  $x$  and  $y$ ,  $x \cdot y$  denotes their inner product.

has the form  $u(x_i) = (\sum_j (a_{ij}x_{ij})^\rho)^{1/\rho}$ , where  $-\infty < \rho < 1$  but  $\rho \neq 0$ . ( $a_{ij} \geq 0$  in all these definitions). As  $\rho$  tends to 1 (resp. 0,  $-\infty$ ), the CES function tends to a linear (resp. Cobb–Douglas, Leontief) function ([1], page 231).

#### Total search problems and the class PPAD

The context of computation of equilibria calls for a complexity analysis conducted within the class *TFNP* of *total search problems*, i.e. problems whose set of solutions is guaranteed to be non empty. Nash Theorem [27] guarantees that the problem of finding a Nash equilibrium in a non-cooperative game in normal form is a total search problem. Arrow and Debreu Theorem [2] gives sufficient conditions under which an exchange economy has an equilibrium. Therefore, under suitable sufficient conditions, the problem of finding a market equilibrium is a total search problem.

However, in general, given an economy expressed in terms of traders' utility functions and initial endowments, an equilibrium does not need to exist. For instance, for economies where the traders have linear utility functions, Gale [15] determined necessary and sufficient conditions for the existence of an equilibrium. Codenotti et al. proved similar results for CES functions [8]. Therefore, unlike what happens with strategic games, the complexity of equilibria for exchange economies falls into the framework of the class *TFNP* only under suitable assumptions. This fact will play a role in some of our results.

An important subclass of *TFNP* is the class *PPAD*, which is the class of total functions whose totality is proven by the following simple combinatorial argument: if a directed graph whose nodes have in-degree and out-degree at most one has a source, it must have a sink. This class was introduced by Papadimitriou [29]. It captures a wealth of equilibrium problems, e.g., the market equilibrium problem as well as Nash equilibria for  $n$ -player games. Problems complete for this class include a (suitably defined) computational version of the Brouwer Fixed Point Theorem.

The computational complexity of computing a Nash equilibrium for two-player nonzero sum games have been the subject of recent investigations. Contrary to previous conjectures, the problem has been proven *PPAD*-complete by Chen and Deng [4]. Chen and Deng's result came after a sequence of works, where first the *PPAD*-completeness of 4-player games [9,19], and then of 3-player games [3,10] were proven. Therefore the correspondence proven in this paper shows that the problem of computing a market equilibrium, when confined to the restrictive scenario of a special family of Leontief economies, is *PPAD*-complete.

#### The game-market correspondence

We consider exchange economies where  $\ell$ , the number of traders, is equal to the number of goods, and the  $i$ -th trader has an initial endowment given by one unit of the  $i$ -th good. The traders have a Leontief (or fixed-proportion) utility function, which describes their goal of getting a bundle of goods in proportions determined by  $\ell$  given parameters.

Given an arbitrary bimatrix game, specified by a pair of  $n \times m$  matrices  $A$  and  $B$ , with positive entries, we construct a Leontief exchange economy with  $n + m$  traders and  $n + m$  goods as follows.

Trader  $i$  has an initial endowment consisting of one unit of good  $i$ , for  $i = 1, \dots, n + m$ . Traders indexed by any  $j \in \{1, \dots, n\}$  receive some utility only from goods  $k \in \{n + 1, \dots, n + m\}$ , and this utility is specified by parameters corresponding to the entries of the matrix  $B$ . More precisely, the proportions in which the  $j$ -th trader wants the goods are specified by the entries on the  $j$ -th row of  $B$ . Viceversa, traders indexed by any  $k \in \{n + 1, \dots, n + m\}$  receive some utility only from goods  $j \in \{1, \dots, n\}$ . In this case, the proportions in which the  $k$ -th trader wants the goods are specified by the entries on the  $k$ -th column of  $A$ .

In the economy above, we can partition the traders in two groups, which bring to the market disjoint sets of goods, and are only interested in the goods brought by the group they do not belong to.

We show that the Nash equilibria of any bimatrix game  $(A, B)$  are in one-to-one correspondence with the market equilibria of such an economy.

#### Applications: NP-hardness results

This one-to-one correspondence allows us to import the results of Gilboa and Zemel [18] on the *NP*-hardness of some computational problems connected with Nash equilibria, and show, among other results, that saying whether there is more than one equilibrium in an exchange economy is *NP*-hard. The latter problem is relevant for applied work, where the uniqueness question is of fundamental importance.

We then prove that saying whether a Leontief exchange economy has an equilibrium is *NP*-hard. More precisely, we construct an economy where the traders have Leontief utility functions, and such that saying whether an equilibrium exists is *NP*-hard. Once again, note that this result does not contradict what is shown in [29], where the market equilibrium problem (both in the version where the input is expressed in terms of utilities and endowments, and in that in terms of excess demand functions) is put in the class *PPAD*. Indeed such a result assumes standard sufficient conditions which guarantee existence by either Kakutani's or Brouwer's fixed point theorem [2].

#### Relation to other work

An important recent advance in computational complexity shows that finding a Nash equilibrium for nonzero sum bimatrix games is complete for the class *PPAD* [4].

The correspondence established in this paper implies that any algorithm which computes a Nash equilibrium for a bimatrix game computes a market equilibrium for a special Leontief economy, and, viceversa, any algorithm for the market equilibrium with Leontief utility functions must have the ability to compute a Nash equilibrium for a bimatrix game.

Therefore the market equilibrium problem is hard for the class *PPAD*, even when restricted to a special Leontief exchange economy.<sup>3</sup>

This fact significantly enhances the current understanding of the problem of computing market equilibria, and clarifies the extent to which there is room for efficient algorithms.

Polynomial time algorithms (or approximation schemes) are only known for markets whose demand function satisfies suitable conditions which guarantee that the set of equilibria is convex [5–7,11,16,17,22,23,28,31].

Roughly speaking, these results take advantage, explicitly or implicitly, of settings where the market's reaction to price changes is *well-behaved* either because the market demand retains some properties of the individual demands or thanks to the special structure of the individual utility functions (e.g., linear, Cobb–Douglas, CES in a certain range of its defining parameter, the elasticity of substitution).

In contrast, exchange economies with *CES utility functions* outside the range studied in [8], i.e. for  $\rho < -1$ , admit multiple disconnected equilibria [20], and no efficient algorithm is known to handle this setting.

A Leontief utility function describes the behavior of an *extreme* CES consumer, i.e.,  $\rho \rightarrow -\infty$ .

Note that CES utility functions with  $\rho < -1$  can approximate Leontief functions, so that efficient algorithms for market equilibria with CES consumers might have implications on the approximability of Nash equilibria for bimatrix games.

Note that the expressive power of Leontief economies is strongly reduced whenever the income of the traders is independent of the prices. Indeed, in such a case, Leontief economies become subject to the aggregation result by Eisenberg [14], and thus an equilibrium can be computed in polynomial time [7].

Finally, our hardness result concerning the existence of an equilibrium in a Leontief exchange economy should be contrasted against the already mentioned characterizations of [8,15] for linear and CES utility functions. These necessary and sufficient conditions boil down to the bi-connectivity of a directed graph, which can be verified in polynomial time.

### Organization of this paper

In Section 2 we define Nash equilibria for bimatrix games as a linear complementarity problem, and specialize the notions of equilibria (and *quasi-equilibria*) for the Leontief economies studied in this paper. In Section 3 we reduce an arbitrary bimatrix game to a special Leontief economy, thus establishing a one-to-one correspondence between the Nash equilibria of the game and the equilibria of the economy.

In Section 4 we first use the one-to-one correspondence stated in Section 3 to import the hardness results of [18] for Nash equilibria in bimatrix games, and get corresponding hardness results for the market equilibrium problem. We then use one of these hardness results to prove that it is *NP*-hard to decide whether a Leontief exchange economy has an equilibrium.

In Section 5 we describe a partial converse of the previous results, by reducing the Leontief economies studied in [32] to bimatrix games.

In Section 6 we mention some related work, and suggest some open questions.

## 2. Games, Markets, and LCP

Let us consider the problem of computing the Nash equilibria for any bimatrix game  $(A, B)$ , where  $A$  and  $B$  are  $m \times n$  matrices, which we assume to be strictly positive without loss of generality.

Recall that a pair of mixed strategies  $(x, y)$  is a Nash equilibrium if  $x^T A y \geq x'^T A y$ , for all stochastic  $m$ -vectors  $x'$ , and  $x^T B y \geq x^T B y'$ , for all stochastic  $n$ -vectors  $y'$ .

Note that  $x^T A y \geq x'^T A y$  implies that  $x^T A y \geq \sum_j a_{ij} y_j$ ,  $i = 1, 2, \dots, m$ , and, similarly,  $x^T B y \geq x^T B y'$  implies that  $x^T B y \geq \sum_i b_{ij} x_i$ ,  $j = 1, 2, \dots, n$ .

Since  $A$  and  $B$  are strictly positive, we have that  $x^T A y$  and  $x^T B y$  are strictly positive. We can thus introduce the vectors  $u$  and  $v$ , whose  $i$ -th entries are  $u_i = \frac{y_i}{x^T A y}$ , and  $v_i = \frac{x_i}{x^T B y}$ , respectively.

In terms of  $u$  and  $v$ , the inequalities above become

$$\begin{aligned} \sum_i b_{ij} v_i &\leq 1 \quad \forall j \\ \sum_j a_{ij} u_j &\leq 1 \quad \forall i. \end{aligned}$$

<sup>3</sup> After a preliminary version of this paper was published in SODA 2006, Huang and Teng showed that the game-market correspondence can be extended to the approximate setting, and can be used to prove hardness results for the approximation of the market equilibrium [21]

If we now add two vectors of *slack variables*,  $r$  and  $t$ , we can transform these inequalities into equalities, i.e.,

$$\begin{cases} B^T v + t = \mathbf{1}, & t \geq 0 \\ Au + r = \mathbf{1}, & r \geq 0, \end{cases}$$

where  $\mathbf{1}$  denotes the vectors (of appropriate sizes) whose components are all equal to 1.

We now have to add the constraint that characterizes the support of a Nash equilibrium in terms of best responses. In other words, we must express the fact that a pure strategy is played with positive probability if and only if it is a best response. Such property can be written as

$$\begin{aligned} x_i > 0 &\rightarrow \sum_j a_{ij} y_j = x^T A y \\ y_i > 0 &\rightarrow \sum_j b_{ji} x_j = x^T B y. \end{aligned}$$

These conditions translate into the complementary constraints  $r \cdot v = t \cdot u = 0$ .

Putting everything together, we can express the Nash equilibrium conditions as the following linear complementarity problem, which we call LCP1.

LCP1: Find a nonnegative  $w \neq \mathbf{0}$  and a nonnegative  $z$  such that

$$\begin{aligned} Hw + z &= \mathbf{1} \\ w^T z &= 0, \end{aligned}$$

where

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \in \mathbf{R}^{(n+m) \times (n+m)}.$$

Note that the system LCP1 may be equivalently viewed as the problem of finding a nonnegative vector  $0 \neq w \in \mathbf{R}^{n+m}$  such that

$$\sum_j h_{ij} w_j \leq 1 \quad \text{for all } 1 \leq i \leq n+m,$$

and

$$w_i > 0 \rightarrow \sum_j h_{ij} w_j = 1 \quad \text{for all } 1 \leq i \leq n+m,$$

where  $h_{ij}$  denotes the  $(i, j)$ -th entry of the matrix  $H$ .

From Nash Theorem on the existence of a Nash equilibrium [27], it follows that LCP1 has at least one solution  $w$ . Let  $\mathcal{N} = \{j : j \leq n\}$  and  $\mathcal{M} = \{j : n < j \leq n+m\}$ . It is easy to see that  $w_j > 0$  for some action  $j \in \mathcal{N}$  as well as for some action  $j \in \mathcal{M}$ , since each of the players is playing a mixed strategy. In other words, if  $w_i > 0$  and  $i \in \mathcal{N}$ , then there must be at least one  $j \in \mathcal{M}$  such that  $w_j > 0$ ; otherwise,

$$1 = \sum_j h_{ij} w_j = \sum_{j=n+1}^{n+m} h_{ij} w_j = 0$$

which is a contradiction. Similarly,  $w_i > 0$  and  $i \in \mathcal{M}$  imply that there must be at least one  $j \in \mathcal{N}$  such that  $w_j > 0$ .

We now describe a special form of a Leontief exchange economy, the *pairing model* [32], in which there are  $\ell$  traders and  $\ell$  goods. The economy is described by a square matrix  $F$  of size  $\ell$ . The  $j$ -th trader has an initial endowment consisting of one unit of the  $j$ -th good, and has a Leontief utility function

$$u_j(x) = \min_{i: f_{ij} \neq 0} \left\{ \frac{x_i}{f_{ij}} \right\}.$$

An equilibrium for such an economy is given by a nonnegative price vector  $0 \neq \pi \in \mathbf{R}^\ell$  such that

1. For each  $1 \leq j \leq \ell$ ,  $\beta_j = \frac{\pi_j}{\sum_k f_{kj} \pi_k}$  is well-defined, that is,  $\sum_k f_{kj} \pi_k > 0$ .
2. For each good  $1 \leq i \leq \ell$ ,  $\sum_j f_{ij} \beta_j \leq 1$ ; that is, the total trading volume does not exceed the quantity available.

Note that  $\beta_j$  represents the utility value of the optimal bundle of the trader  $j$  at equilibrium, and the optimal bundle itself is  $(f_{1j} \beta_j, \dots, f_{\ell j} \beta_j)$ . Standard arguments imply that if  $\pi_i > 0$ , then in fact  $\sum_j f_{ij} \beta_j = 1$ . Moreover, we also have that  $\pi_j > 0$  if and only if  $\beta_j > 0$ .

A closely related notion is that of a *quasi-equilibrium*. In our setting, a quasi-equilibrium is obtained by replacing condition (1) above by

1'. For each  $1 \leq j \leq \ell$ , there exists  $\beta_j$  such that  $\beta_j(\sum_k f_{kj}\pi_k) = \pi_j$ .

In a quasi-equilibrium, the zero-bundle, corresponding to  $\beta_j = 0$ , is a valid bundle when  $\pi_j = 0$ , even though  $\sum_k f_{kj}\pi_k = 0$ .

Thus the main difference between an equilibrium and a quasi-equilibrium is that in the latter, a trader with zero income is not required to optimize her utility. The reader is referred to the textbook of Mas-Colell et al. [25] for a more systematic development. One standard way to establish sufficient conditions for the existence of an equilibrium is to first use fixed point theorems to establish the existence of a quasi-equilibrium, and then argue that under the sufficient conditions, every quasi-equilibrium is an equilibrium.

A simple example of a (pairing) Leontief economy that has a quasi-equilibrium but no equilibrium is encoded by the matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed it is easy to check that  $\pi = (0, 1, 0)$  is a quasi-equilibrium. On the other hand, condition (1) above for the existence of an equilibrium cannot be satisfied.

### 3. Leontief economies encode bimatrix games

We give a polynomial time computable reduction from any two-player nonzero sum game to the Leontief exchange economies constructed above with the property that the Nash equilibria of the game and the equilibria of the market are in one-to-one correspondence. This shows that the problem of computing Nash equilibria for a bimatrix game is equivalent to that of computing market equilibria for these exchange economies. To prove this result, we follow the approach of Ye [32].

Given an instance of the problem of computing the Nash equilibria for a bimatrix game  $(A, B)$ , where  $A$  and  $B$  are positive  $n \times m$  matrices, we construct an instance of a (pairing) exchange economy with  $(n + m)$  traders and  $(n + m)$  goods that is given by setting  $F = H$ . It is also easy to see that trading needs to occur between some trader  $j \in \mathcal{N}$  and some trader  $j \in \mathcal{M}$ , since traders in  $\mathcal{N}$  are only interested in goods that are brought in by traders in  $\mathcal{M}$ , and viceversa. We call this economy *two-groups Leontief economy*. It easily follows from the definition that at any equilibrium  $\pi$  of the economy, we must have  $\pi_i > 0$  for some  $i \in \mathcal{N}$  as well as for some  $i \in \mathcal{M}$ .

*From the market to the game*

We first prove that any market equilibrium of the two-groups Leontief economy corresponds to a Nash equilibrium in the associated two-player bimatrix game.

**Lemma 1.** *Let  $\beta = (\beta_1, \dots, \beta_{n+m})$  be the vector of the utility values at equilibrium prices  $\pi$  for the two-groups Leontief economy. Then  $\beta$  solves LCP1, and thus it encodes the Nash equilibria of the game described by LCP1.*

**Proof.** At any equilibrium of the market, we have  $\sum_j h_{ij}\beta_j \leq 1$  for each  $1 \leq i \leq n + m$ , and  $\beta_j > 0$  if and only if  $\pi_j > 0$ . Moreover,  $\beta_i > 0 \rightarrow \sum_j h_{ij}\beta_j = 1$ . Thus the  $\beta$ 's from the equilibrium solve the system LCP1 with  $w = \beta$ . Moreover,  $\pi_j$ , and thus  $\beta_j$ , is positive for some  $j$ , so that  $w = \beta \neq 0$ .

*From the game to the market*

We now show that any Nash equilibrium of a bimatrix game corresponds to a market equilibrium of the corresponding two-groups Leontief economy.

**Lemma 2.** *Let  $w \neq 0$ , be any solution to LCP1. Then there exists an equilibrium price vector  $\pi$  such that  $w = (w_1, \dots, w_{n+m})$  is the vector of the utility values at these equilibrium prices for the two-groups Leontief economy.*

**Proof.** Let  $w \neq 0$  be any complementarity solution to LCP1. Partition the index set  $\{1, \dots, n + m\}$  into two sets  $P = \{j : w_j > 0\}$  and  $Z = \{j : w_j = 0\}$ . As we showed before,  $P \cap \mathcal{N} \neq \emptyset$  and  $P \cap \mathcal{M} \neq \emptyset$ .

We claim that there exists  $\pi_j > 0$  for each  $j \in P$  such that  $w_j = \frac{\pi_j}{\sum_{k \in P} h_{kj}\pi_k}$ , or in a different form,  $\sum_{k \in P} h_{kj}w_j\pi_k = \pi_j$ . Let  $H_{PP}$  be the  $|P| \times |P|$  principal submatrix of  $H$  induced by the indices in  $P$ , and  $W_P$  the  $|P| \times |P|$  diagonal matrix whose diagonal contains the  $w$ 's corresponding to  $P$ . Our claim is equivalent to saying that the system  $C\sigma = \sigma$ , where  $C = (H_{PP}W_P)^T$ , has a solution in which all the entries of  $\sigma$  are positive. Note that each column of  $C$  sums to one: this follows because  $i \in P \rightarrow w_i > 0$  and

$$w_i > 0 \rightarrow \sum_{j \in P} h_{ij}w_j = \sum_j h_{ij}w_j = 1.$$



Moreover,

$$C = \begin{pmatrix} 0 & D \\ E^T & 0 \end{pmatrix},$$

where  $E$  and  $D$  are  $(|P| - l) \times l$  matrices, for some  $1 \leq l \leq |P| - 1$ . The bounds on  $l$  follow from the fact that  $P \cap \mathcal{N} \neq \emptyset$  and  $P \cap \mathcal{M} \neq \emptyset$ .

The existence of such a positive solution to  $C\sigma = \sigma$  follows from [Proposition 3](#) below.

We have established our claim that there exists  $\pi_j > 0$  for each  $j \in P$  such that

$$w_j = \frac{\pi_j}{\sum_{k \in P} h_{kj} \pi_k}.$$

Set  $\pi_j = 0$  for  $j \in Z$ . We now argue that  $\pi$  is an equilibrium.

Note that for  $j \in P$ , we have

$$w_j = \frac{\pi_j}{\sum_{k \in P} h_{kj} \pi_k} = \frac{\pi_j}{\sum_k h_{kj} \pi_k}.$$

For  $j \in Z$ , observe that  $\sum_k h_{kj} \pi_k > 0$ . This is because there exists  $k \in P$  such that  $h_{kj} > 0$ , since  $P$  contains elements from both  $\mathcal{N}$  and  $\mathcal{M}$ . For this  $k$ , we have  $h_{kj} \pi_k > 0$ . Therefore,

$$w_j = \frac{\pi_j}{\sum_{k \in P} h_{kj} \pi_k} = \frac{\pi_j}{\sum_k h_{kj} \pi_k} = 0.$$

Moreover, we have, for each good  $1 \leq i \leq n + m$ ,  $\sum_j h_{ij} w_j \leq 1$ , since  $w$  is a solution of LCP1. Thus both the conditions for an equilibrium are fulfilled, with the  $w_i$ 's playing the role of the  $\beta_i$ 's.

**Proposition 3.** *The linear system  $C\sigma = \sigma$  has a positive solution.*

**Proof.** Consider the matrix

$$C^2 = \begin{pmatrix} DE^T & 0 \\ 0 & E^T D \end{pmatrix}.$$

Notice that both  $DE^T$  and  $E^T D$  are column stochastic, because  $C$  and hence  $D$  and  $E^T$  are column stochastic. This fact, together with the positivity of both  $DE^T$  and  $E^T D$ , implies that the system  $C^2 z = z$  has a positive solution.

We can now write  $(C^2 - I)z = 0$  as  $(C - I)(C + I)z = 0$ . Consider the vector  $\sigma = (C + I)z$ . Clearly  $\sigma$  has all positive components, if  $z$  has. Also  $(C - I)\sigma = 0$  or  $C\sigma = \sigma$ .

Note that [Proposition 3](#) implies that  $C$  is irreducible besides column-stochastic, so that  $\sigma$  is in fact the unique Perron–Frobenius eigenvector of  $C$  (see, for example, [24], p. 141). Consequently, we observe that there is precisely one equilibrium price vector  $\pi$ , the one we have constructed above, that corresponds to the utility vector  $w$ . This follows because we must have  $\pi_j > 0$  if and only if  $w_j > 0$ . Thus  $\pi_j = 0$  for  $j \in Z$ ,  $\pi_j > 0$  for  $j \in P$ , and thus the unique positive solution of  $C\sigma = \sigma$  gives the only possible values for the prices for goods in  $P$ . From the definition, it follows that there is a unique utility vector corresponding to an equilibrium price vector.

The following theorem summarizes the results of this section.

**Theorem 4.** *Let  $(A, B)$  denote an arbitrary bimatrix game, where we assume, w.l.o.g., that the entries of the matrices  $A$  and  $B$  are all positive. Let the columns of*

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$$

*describe the utility parameters of the traders in a two-groups Leontief economy. There is a one-to-one correspondence between the Nash equilibria of the game  $(A, B)$  and the market equilibria of the two-groups Leontief economy. Furthermore, the correspondence has the property that a strategy is played with positive probability at a Nash equilibrium if and only if the good held by the corresponding trader has a positive price at the corresponding market equilibrium.*

**Corollary 5.** *The problem of finding an equilibrium for a two-groups Leontief economy is PPAD-complete.*

#### 4. Hardness results

Well known sufficient conditions guarantee that an equilibrium for an exchange economy does exist (see, e.g., [25] Section 17C). Under such assumptions, its equivalence to fixed point problems follows from the combination of two results: a simple transformation introduced by Uzawa [30], which maps any continuous function into an excess demand function, inducing a one-to-one correspondence between the fixed points of the function and the equilibria, and the Sonnenschein-Mantel-Debreu Theorem (see [25], pp. 598–606) which states the essentially arbitrary nature of the market excess demand function.

**Theorem 4** shows that there is a one-to-one correspondence between two-groups Leontief economies and bimatrix games. Combining this theorem with the *NP*-hardness results of Gilboa and Zemel for some questions related to Nash equilibria [18], we show hardness results for Leontief economies.

One of these hardness results pertains the existence of an equilibrium where the prices of some prescribed goods are positive. This specific hardness result allows us to construct a Leontief exchange economy for which an equilibrium exists if and only if in another Leontief economy there is an equilibrium where the prices of some prescribed goods are positive. This correspondence proves that it is *NP*-hard to test for existence.

Note that in general the equilibria of Leontief exchange economies can be irrational ([7], Section 3) so that the existential problem might not belong to *NP*, and we thus talk of *NP*-hardness as opposed to *NP*-completeness.

##### 4.1. Uniqueness and equilibria with additional properties

Gilboa and Zemel [18] proved a number of hardness results related to the computation of Nash equilibria (NE) for finite games in normal form. Since the NE for games with more than two players can be irrational, these results have been formulated in terms of *NP*-hardness for multi-player games, while they can be expressed in terms of *NP*-completeness for two-player games.

Given a two-player game  $G$  in normal form, i.e. expressed as a bimatrix game, consider the following problems:

1. *NE uniqueness*: Given  $G$ , does there exist a unique NE in  $G$ ?
2. *NE in a subset*: Given  $G$ , and a subset of strategies  $T_i$  for each player  $i$ , is there a NE where all the strategies outside  $T_i$  are played with probability zero?
3. *NE containing a subset*: Given  $G$ , and a subset of strategies  $T_i$  for each player  $i$ , is there a NE where all the strategies in  $T_i$  are played with positive probability?
4. *NE maximal support*: Given  $G$  and an integer  $r \geq 1$ , does there exist a NE in  $G$  such that each player uses at least  $r$  strategies with positive probability?
5. *NE minimal support*: Given  $G$  and an integer  $r \geq 1$ , does there exist a NE in  $G$  such that each player uses at most  $r$  strategies with positive probability?

Gilboa and Zemel showed that

1. *NE uniqueness* is co-*NP* complete;
2. *NE in a subset*, *NE containing a subset*, *NE maximal support*, and *NE minimal support* are *NP*-complete.

Combining the above results with **Theorem 4**, we get the following theorem.

**Theorem 6.** *Given an exchange economy, where each trader is specified by an initial endowment and a Leontief utility function, such that the economy has at least one equilibrium, the following problems are NP-hard:*

1. *Is there more than one equilibrium?*
2. *Is there an equilibrium where the prices of a given set of goods are positive?*

**Proof.** The results use the reduction of **Theorem 4**, which, together with Nash Theorem on the existence of a Nash equilibrium, tells us that the Leontief economy constructed by the reduction always has an equilibrium.

1. The *NP*-hardness follows from the co*NP*-completeness of *NE uniqueness*, and from the one-to-one correspondence of **Theorem 4**. We also note that the construction of Gilboa and Zemel [18] for *NE uniqueness* yields games with a finite number of equilibria.
2. The *NP*-hardness follows from the *NP*-completeness of *NE containing a subset*, and from **Theorem 4**.



#### 4.2. Existence of an equilibrium

We now give a reduction from statement (2) of [Theorem 6](#) to show that the problem of deciding whether a Leontief exchange economy has an equilibrium is NP-hard.

**Theorem 7.** *It is NP-hard to decide whether a Leontief exchange economy has an equilibrium.*

**Proof.** The reduction is from [Theorem 6](#) (2). Suppose  $M$  is an instance of an economy with  $n$  traders and goods, and we want to know if there is an equilibrium with goods  $1, \dots, k$  priced positively. We construct an economy  $M'$  with  $k$  additional traders and goods: for  $1 \leq j \leq k$ , the  $(n+j)$ -th trader brings in one unit of the  $(n+j)$ -th good and wants just the  $j$ -th good.

We argue that  $M'$  has an equilibrium if and only if  $M$  has an equilibrium with goods  $1, \dots, k$  priced positively.

Suppose  $M$  has an equilibrium in which goods  $1, \dots, k$  are priced positively. Then this can be extended to an equilibrium of  $M'$  by setting the prices of goods  $n+1, \dots, n+k$  to be 0, and giving the  $(n+j)$ -th trader 0 utility (and 0 units of good  $j$ ). It is evident that condition (1) for an equilibrium holds for the  $(n+j)$ -th trader, since the  $j$ -th good is priced positively. Condition (2) also holds.

Consider now an equilibrium for  $M'$ . For  $1 \leq j \leq k$ , it can be seen from Walras' Law that the price of the  $(n+j)$ -th good must be zero, since nobody wants this good. For condition (1) to hold for the  $(n+j)$ -th trader, it must be that the  $j$ -th good is priced positively. It follows that the prices of the first  $n$  goods, together with the optimal bundles of the first  $n$  traders, constitutes an equilibrium for the original economy  $M$  in which the prices of goods  $1, \dots, k$  are positive.

We have proved that  $M'$  has an equilibrium if and only if  $M$  has an equilibrium with goods  $1, \dots, k$  priced positively.  $M'$  can clearly be constructed from  $M$  in polynomial time.

Notice that the reduction can be easily modified, if needed, to ensure that each good in  $M'$  is desired by some trader. (We simply make the  $(n+j)$ -th trader want both the  $(n+j)$ -th good and the  $j$ -th good in the ratio 1:2.)

#### 5. Bimatrix games encode the (pairing) Leontief economy

In this section, we establish a partial converse of the result of [Section 3](#). We will show that bimatrix games encode a special case of the pairing Leontief economies. In this setting, there are  $n$  traders and  $n$  goods. The  $j$ -th trader has an initial endowment given by one unit of the  $j$ -th good, and has a Leontief utility function

$$u_j(x) = \min_i \left\{ \frac{x_i}{a_{ij}} \right\},$$

where  $a_{ij} > 0$ , for all  $i, j$ .

We will show that finding equilibrium prices for the economy above is equivalent to finding equilibria for the bimatrix game  $(A, I)$ , where  $I$  denotes the identity matrix.

Consider the following linear complementarity problem, which we call LCP(A): Find  $w \neq 0$  such that

$$\begin{aligned} Aw &\leq \mathbf{1} \\ w_i > 0 &\rightarrow (Aw)_i = 1 \\ w &\geq 0. \end{aligned}$$

**Theorem 8.** *Let  $y$  be a nonnegative and nonzero  $n$ -vector. Let  $x$  be the  $n$ -vector such that  $x_i = 1$  if  $y_i > 0$ , and  $x_i = 0$  if  $y_i = 0$ . Let  $\tilde{x} = \alpha x$ , with  $\alpha = \frac{1}{x_1 + \dots + x_n}$ , and  $\tilde{y} = \beta y$ , with  $\beta = \frac{1}{y_1 + \dots + y_n}$ . The vector  $y$  is a solution to LCP(A) if and only if  $(\tilde{x}, \tilde{y})$  is a Nash equilibrium for the bimatrix game  $(A, I)$ .*

**Proof.** The linear complementarity formulation for the Nash equilibria of the game  $(A, I)$  (see [Section 2](#)) consists of finding nonnegative, and not both zero, solutions  $z$  and  $w$  to the system

$$\begin{aligned} Az &\leq \mathbf{1} \\ w &\leq \mathbf{1} \\ w_i > 0 &\rightarrow (Az)_i = 1 \\ z_i > 0 &\rightarrow w_i = 1, \end{aligned}$$

which we call LCP(A, I).

Since  $z_i > 0$  if and only if  $w_i > 0$ , we can conclude that if  $(z, w)$  solves LCP(A, I), then  $z$  solves LCP(A).

Conversely, it is easy to see that, if a vector  $z$  solves LCP(A), then the pair of vectors  $(z, w)$ , where  $w$  be the  $n$ -vector such that  $w_i = 1$  if  $z_i > 0$ , and  $w_i = 0$  if  $z_i = 0$ , solve LCP(A, I).

We now argue that any nonzero solution to the complementarity problem LCP(A), or equivalently any Nash equilibrium of the game  $(A, I)$ , corresponds to an equilibrium of the Leontief economy.

**Theorem 9.** For any nonzero solution  $w$  of  $LCP(A)$  with a positive matrix  $A$ , there is an equilibrium price  $\pi$  such that the utility value of player  $i$  at  $\pi$  is  $w_i$ . Moreover, given  $w$ ,  $\pi$  can be computed in polynomial time.

**Proof.** The proof of this theorem is implied by [32]. Let  $P = \{j : w_j > 0\}$ , and  $Z = \{j : w_j = 0\}$ . Then consider the stochastic matrix  $A_{PP}D(w_P)$ , where  $A_{PP}$  is  $|P| \times |P|$  principal submatrix of  $A$  induced by the indices in  $P$ ,  $D(w_P)$  is the diagonal matrix whose entries are  $w_j$ ,  $j \in P$ . Since  $A_{PP}D(w_P) > 0$ , it has a positive left eigenvector  $\pi_P > 0$ . Let  $\pi_j = 0$  for  $j \in Z$ .

Since for some  $i$ ,  $w_i > 0$ ,  $P$  is non-empty and therefore  $\pi$  is also nonzero. Furthermore, it is very easy to see that:

1. For every  $1 \leq i \leq n$ ,  $\sum_{j=1}^n a_{ij}w_j \leq 1$
2.  $w_i > 0 \rightarrow \sum_{j=1}^n a_{ij}w_j = 1$ .

Therefore,  $w$  is an allocation supported by the equilibrium price vector  $\pi$ .

It is straightforward to see that any equilibrium of the pairing Leontief economy yields a Nash equilibrium of the game  $(A, I)$ .

Note that McLennan and Tourky [26] have also shown that finding a Nash equilibrium of a bimatrix game is polynomial time equivalent to finding a solution to an instance of  $LCP(A)$ . Their approach provides an alternative way to derive the equivalence results of this paper.

It is worthwhile to point out that, while the reduction in Theorem 4 is from arbitrary bimatrix games, the reduction in this section is from only a special family of Leontief economies. As in bimatrix games, the equilibrium points of the pairing Leontief economies are rational numbers [32]. However, in the case where the endowments of the buyers are unrestricted, Eaves [12] gives an example showing that equilibrium points could be irrational. This suggests that there is no natural linear complementarity formulation for general Leontief exchange economies.

Furthermore, we have assumed that the entries of the utility matrix  $A$  are positive. This restriction is necessary because if some entries of  $A$  are zero,  $A_{PP}$  may be reducible and a strictly positive left eigenvector  $\pi_P$  may not exist. This shows a subtle difference in the structure of equilibria in these two settings despite their similar linear complementarity programs. It is easy to see that adding a constant to all the entries of a matrix corresponding to a game does not change its equilibrium points, but adding a constant to all entries of the utility matrix of a Leontief economy might change the set of equilibria.

## 6. Concluding remarks

In this paper, we have introduced a correspondence between exchange economies and bimatrix games, and analyzed some related computational consequences.

Prior to this work, Eaves has shown in [12] that the equilibrium in exchange economies with Cobb–Douglas utility functions can be obtained as the solution to a special linear programming problem. Because of the well known equivalence between zero-sum games and linear programming (due to Von Neumann Minimax Theorem), we have that Cobb–Douglas exchange economies can be coded as special two-player zero-sum games, and thus can be reduced to the Leontief exchange economies studied in this paper.

In another piece of work, Eaves [13] has shown that the equilibria for linear exchange economies can be obtained as solutions to a linear complementarity problem. It would be interesting to see if this complementarity problem can be expressed in the form of  $LCP1$ , where  $H$  is a nonnegative matrix. If so, the technique in Section 5 can be used to reduce it to a bimatrix game, and hence to a Leontief exchange economy.

The reductions from linear or Cobb–Douglas economies to Leontief economies would be from ‘easy’ to ‘hard’ problems, but they would nevertheless be an interesting exercise.

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